

KINEMATIC FORMULAS FOR SETS DEFINED BY DIFFERENCES OF CONVEX FUNCTIONS

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ABSTRACT. The class $WDC(M)$ consists of all subsets of a smooth manifold M that may be expressed in local coordinates as certain sublevel sets of DC (differences of convex) functions. If M is Riemannian and G is a group of isometries acting transitively on the sphere bundle SM , we define the invariant curvature measures of compact WDC subsets of M , and show that pairs of such subsets are subject to the array of kinematic formulas known to apply to smoother sets. Restricting to the case $(M, G) = (\mathbb{R}^n, SO(n))$, this extends and subsumes Federer's theory of sets with positive reach in an essential way. The key technical point is equivalent to a sharpening of a classical theorem of Ewald, Larman, and Rogers characterizing the dimension of the set of directions of line segments lying in the boundary of a given convex body.

1. INTRODUCTION

The classical *principal kinematic formula* (PKF) expresses, in terms of geometric quantities (intrinsic volumes) associated separately to compact subsets $A, B \subset \mathbb{R}^d$, the integral of the Euler characteristic of the intersection $A \cap \gamma B$ over $\gamma \in SO(d)$. However, it is necessary to restrict A, B to have “reasonable” smoothness: the original framework of Blaschke assumed A, B to be convex, and subsequently Santaló and Chern [26, 5] showed that the formula holds when A, B are smooth domains. Both these cases were subsumed by the theory of Federer [8], treating the case of sets A, B of *positive reach*. This theory has represented the state of the art for many years: the extensions by Hadwiger to sets from the “convex ring” (the class of finite unions of compact convex bodies), and in [29] to the class U_{PR} of finite unions of sets with positive reach in general position, both rely on the analysis of the convex/positive reach case; and the extension of [12] to subanalytic sets relies on the very special finiteness properties that these sets enjoy (in fact the methods there apply also to sets definable with respect to any given o-minimal structure [27]).

It is natural to ask to characterize precisely the minimal amount of smoothness needed to ensure that the PKF holds. This question turns out to be subtle and elusive, and indeed it appears to evade all classical smoothness classes. The paper [12] attempted to formulate an answer using a notion of smoothness arising from the apparatus of the proof of the PKF itself. The basic object of interest is the *normal cycle* $N(A)$ of $A \subset \mathbb{R}^d$, viz. an integral current associated to a singular subspace A that stands in for the manifold of unit normals of a smooth set A . Closely related is the *differential cycle* $\mathbb{D}(f)$ of a nonsmooth function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, which is an integral current that stands in for the graph of the differential of a smooth f . A function f

Date: October 14, 2015.

2010 Mathematics Subject Classification. 53C65, 52A20.

Key words and phrases. Curvature measures, kinematic formula, positive reach, DC functions, WDC sets, normal cycle, Monge-Ampère functions.

JHGF was supported by NSF grant DMS-1406252. The second author is a junior researcher in the University Centre for Mathematical Modelling, Applied Analysis and Computational Mathematics (MathMAC). The second and third authors were also supported by grant GAČR 15-08218S.

that admits such a differential cycle is called a *Monge-Ampère (MA) function*. The general theory of [12] posits that a set A subject to the PKF should be given as a sublevel set of an MA function at a *weakly regular* value (cf. Definition 2.5 below). Unfortunately, the theory found only limited success: in order to prove the PKF for pairs of such sets it was necessary to introduce additional ad hoc hypotheses on the supports of $N(A)$ and $\mathbb{D}(f)$ (viz. the hypotheses on $\text{nor}(f, 0)$, $\text{nor}(g, 0)$ in Theorem 2.2.1 of [12]).

The main point of the present paper is to show that the general scheme of [12] works completely, without these ad hoc devices, in the case of the WDC sets introduced in [22]. In this last work it was shown first of all that any DC function f (i.e. a function expressible locally as $f = g - h$, where g, h are convex) is MA. A set A is WDC if it may be expressed as a sublevel set of a DC function f at a weakly regular value. In the present paper, by sharpening a theorem of Ewald, Larman, and Rogers [6], and using a construction of Pavlica and Zajíček [21], we show that the unwanted ad hoc hypotheses are always fulfilled in this setting.

Using a characterization of sets with positive reach due to Kleinjohann and Bangert (Theorem 2.9 below), it is easy to see that any set with positive reach is a WDC set. Since WDC is closed under finite unions and intersections in general position, it follows that any U_{PR} set (in the original sense of [29]) is also WDC. Thus the theory developed here subsumes that of [8, 29], and indeed ventures well beyond it, covering for example in a systematic way also the case of general convex hypersurfaces.

1.1. Plan of the paper. In fact there are many kinematic formulas beyond the PKF, which may be treated together in terms of integration of invariant forms over the normal cycle. It is known ([11], [1]) that if M is Riemannian and admits a Lie group G of isometries that acts transitively on the tangent sphere bundle SM (i.e. (M, G) is a **Riemannian isotropic space**) then kinematic formulas exist for pairs of subsets of M belonging to any of the classical integral geometric regularity classes above [11]. We will carry out our analysis of WDC sets in this same general context.

Section 2 is devoted to notions that will figure importantly in the subsequent discussion. In particular we recall from [22] the definitions of DC and MA functions and the key inclusion $\text{DC} \subset \text{MA}$. We give a new result (Theorem 2.16) describing the differential cycle of the sum of two MA functions in general position on a homogeneous space; this formula is key to the subsequent proof of the kinematic formulas. We recall also the definition of a WDC set A as the level set $A = f^{-1}(0)$ of a nonnegative DC function f for which 0 is a weakly regular value. In this case we say that f is a DC **aura** for A , and we recall the main result of [22], giving the construction of the conormal cycle of A in terms of f and showing that it is independent of the choice of aura f .

The main result (Theorem 3.1) of Section 3 states that in this setting the sets $\text{nor}_\varepsilon(f)$, defined as the set of all elements of the sphere bundle SM that arise as normalized Clarke differentials of f at $f = 0$, has locally finite $(d - 1)$ -dimensional Minkowski content, where $d = \dim M$. Since Minkowski content—unlike Hausdorff measure—behaves well under Cartesian products, this fact is the key to establishing the support condition needed to prove the kinematic formulas. The proof of Theorem 3.1 relies ultimately on a fundamental construction of Pavlica and Zajíček [21] relating the support elements of the graph of a DC function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ to the set of support hyperplanes P common to two different convex subsets $A, B \subset \mathbb{R}^{d+1}$. A lemma of Ewald-Larman-Rogers [6] states that the latter set has the expected Hausdorff dimension; this is the key lemma for their well-known theorem stating that the set of directions of line segments lying in the boundary of a given convex

body in \mathbb{R}^d has Hausdorff dimension at most $d - 2$. As shown in [22], this argument is enough to show that the set of tangent hyperplanes P to the graph of the DC function f has the expected Hausdorff dimension d , which in turn is enough to show that a WDC set admits a normal cycle in the sense of [12], Theorem 3.2.

However, necessary for our main result (Theorem B below) is the stronger assertion that the set of pairs (x, P) such that P is a tangent plane for the graph of f at $(x, f(x))$ has the same dimension d , and moreover that this dimension may be evaluated in the sense of Minkowski content. This result (Lemma 3.5) follows from a refinement of the Pavlica-Zajíček result: given convex $A, B \subset \mathbb{R}^d$, the set of pairs $(x - y, P)$ such that P is a support plane both for A at x and for B at y has Minkowski dimension d . As a byproduct of this analysis we also arrive at the following enhanced version of the theorem of Ewald-Larman-Rogers:

Theorem A. *Let $K \subset \mathbb{R}^d$ be closed and convex. Denote by T_K the set of pairs $(v, w) \in S^{d-1} \times S^{d-1}$ with the property that there exists a nondegenerate segment $\tau \subset \partial K$ with direction v and lying in a supporting hyperplane of K with outward normal direction w . Then T_K has σ -finite $(d - 2)$ -dimensional Minkowski content.*

In Section 4 we prove our main theorem, that the kinematic formulas described in [11] hold for pairs of WDC sets in an isotropic space (M, G) (classical work of Hartman [14] implies that the space of DC functions is stabilized by diffeomorphisms, hence this notion, and the notion of WDC set, make sense on a smooth manifold). In order to state this precisely, let d be the dimension of M . Since (M, G) is Riemannian isotropic, it is clear that the space of G -invariant differential forms on SM is canonically isomorphic to the subspace of $G_{\bar{o}}$ -invariant elements of the exterior algebra $\bigwedge^* T_{\bar{o}}^*(SM)$, where $G_{\bar{o}} \subset G$ is the stabilizer of an arbitrary point $\bar{o} \in SM$. In particular, this space has finite dimension. Each such form β of degree $d - 1$ gives rise to a G -invariant **curvature measure** on M , i.e. an object that associates to each WDC set $A \subset M$ the signed measure $\Phi_\beta(A, \cdot)$ given by

$$\Phi_\beta(A, E) := \int_{N(A) \perp \pi^{-1}(E)} \beta,$$

where $\pi : SM \rightarrow M$ is the projection. This measure may alternatively be viewed as a linear functional on the space of bounded Borel measurable functions ϕ given by

$$\Phi_\beta(A, \phi) := \int_{N(A)} \pi^* \phi \wedge \beta.$$

Denote the space of all such Φ_β by \mathcal{C}^G . For $\Phi, \Psi \in \mathcal{C}^G$ and $A, B \in \text{WDC}(M)$ we put

$$(\Phi \otimes \Psi)(A, \phi; B, \psi) := \Phi(A, \phi) \Psi(B, \psi)$$

and extend to all of $\mathcal{C}^G \otimes \mathcal{C}^G$ by bilinearity.

Theorem B. *Let (M, G) be a Riemannian isotropic space, and put $d\gamma$ for the Haar measure on G that projects to the Riemannian volume of M .*

- (1) *If $A, B \in \text{WDC}(M)$ then $A \cap \gamma B \in \text{WDC}(M)$ for a.e. $\gamma \in G$.*
- (2) *There exists a linear map*

$$K : \mathcal{C}^G \rightarrow \mathcal{C}^G \otimes \mathcal{C}^G$$

such that, for any compact $A, B \in \text{WDC}(M)$ and bounded Borel measurable functions $\phi, \psi : M \rightarrow \mathbb{R}$

$$(1.1) \quad \int_G \Phi(A \cap \gamma B, \phi \cdot (\psi \circ \gamma^{-1})) d\gamma = K(\Phi)(A, \phi; B, \psi).$$

Conclusion (2) may be restated in more prosaic terms as follows:

Let β_1, \dots, β_N be a basis for the vector space of G -invariant differential forms of degree $d-1$ on SM . Then there exist constants c_{ij}^k such that given any compact sets $A, B \in \text{WDC}(M)$ and bounded Borel measurable functions $\phi, \psi : M \rightarrow \mathbb{R}$, then

$$(1.2) \quad \begin{aligned} \int_G \left(\int_{N(A \cap \gamma^{-1}B)} \pi^*(\phi \cdot (\psi \circ \gamma^{-1})) \wedge \beta_k \right) d\gamma &= \sum_{i,j} c_{ij}^k \int_{N(A)} \pi^* \phi \wedge \beta_i \int_{N(B)} \pi^* \psi \wedge \beta_j \\ &+ \int_A \phi \cdot \int_{N(B)} \pi^* \psi \wedge \beta_k \\ &+ \int_B \psi \cdot \int_{N(A)} \pi^* \phi \wedge \beta_k, \quad k = 1, \dots, N. \end{aligned}$$

For A, B of positive reach and $(M, G) = (\mathbb{R}^n, \overline{SO(n)})$ this is the classical kinematic formula of Federer [8, Theorem 6.11].

1.2. Acknowledgments. It is a pleasure to thank L. Zajíček for helpful conversations.

2. BACKGROUND

2.1. Generalities and notation.

2.1.1. General concepts. The volume of the unit ball in \mathbb{R}^d is denoted

$$\omega_d := \pi^{\frac{d}{2}} / \Gamma\left(\frac{d}{2} + 1\right).$$

We put Π_L for the orthogonal projection onto an affine subspace $L \subset \mathbb{R}^d$.

We write $A \subset\subset B$ to mean that A is a compact subset of B .

If E is a Cartesian product $A \times B$ or $B \times A$, then we use $\pi_A : E \rightarrow A$ to denote the projection.

2.1.2. Currents. We generally follow the notation and terminology of [9], with some minor deviations. A **current** T of dimension k in the smooth manifold M is a linear functional T on the space $\Omega_c^k(M)$ of compactly supported smooth differential forms on M , continuous with respect to C^∞ convergence with uniformly compact support. The **support** of T is denoted $\text{spt } T$. The pairing of a current T against a differential form ϕ will usually be denoted $\int_T \phi$. A current T is **representable by integration** if there exist a Radon measure $\|T\|$ and a Borel measurable k -vector field \vec{T} on M such that

$$\int_T \phi = \int_M \langle \vec{T}_x, \phi_x \rangle d\|T\|x, \quad \phi \in \Omega_c^k(M).$$

If M is an oriented C^1 manifold, we will often conflate a measurable subset $A \subset M$ with the current defined by integration over A .

Among all currents the group $\mathbb{I}_k(M)$ of **integral currents** ([9], 4.1.24) of dimension k in manifold M enjoys many special properties. Any integral current T may be pushed forward by a proper Lipschitz map $f : M \rightarrow N$ to yield a current $f_*T \in \mathbb{I}_k(N)$ via the formula

$$\int_{f_*T} \phi := \int_T f^* \phi.$$

We will rely heavily on the Federer-Fleming theory of **slicing**, described in detail in Section 4.3 of [9]. The slice of T by such f at $y \in N$ is denoted $\langle T, f, y \rangle$. If T is given by integration over a smooth submanifold V , and y is a regular value of

f , then this slice is simply the current given by integration over the appropriately oriented intersection of V with $f^{-1}(y)$. If $S \in \mathbb{I}_k(Y)$ for some manifold Y , recall that the conventions of [9], Section 4.3, imply that for $y \in N$

$$(2.1) \quad \begin{aligned} \pi_{Y*} \langle N \times S, \pi_N, y \rangle &= S, \\ \pi_{Y*} \langle S \times N, \pi_N, y \rangle &= (-1)^{k \dim N} S, \end{aligned}$$

where the Cartesian product of currents is defined as in [9], 4.1.8. One particularly important fact is **commutativity of pushforward and slicing** ([9], Theorem 4.3.2 (7)): if

$$(2.2) \quad \begin{aligned} M &\xrightarrow{g} N \xrightarrow{h} L \\ \text{are Lipschitz maps, and } T &\in \mathbb{I}_*(M) \text{ then for a.e. } y \in L \\ g_* \langle T, h \circ g, y \rangle &= \langle g_* T, h, y \rangle. \end{aligned}$$

The boundary of a current T is denoted by ∂T , and defined by the formula

$$\int_{\partial T} \phi := \int_T d\phi,$$

and the boundary of a Cartesian product is

$$\partial(S \times T) = \partial S \times T + (-1)^{\dim S} S \times \partial T.$$

Slicing behaves naturally with respect to the boundary operation: if $g : M \rightarrow N$, $\dim N = n$ then for a.e. $y \in N$ (cf. [9], p. 437)

$$(2.3) \quad \partial \langle T, g, y \rangle = (-1)^n \langle \partial T, g, y \rangle.$$

If T is a current of dimension k and ϕ is a differential form of degree $j \leq k$ then $T \llcorner \phi$ is the current of dimension $k - j$ defined by

$$\int_{T \llcorner \phi} \psi = (T \llcorner \phi)(\psi) := \int_T \phi \wedge \psi.$$

If T is integral (in particular, representable by integration), the formula above makes sense even when ϕ is merely bounded and Borel measurable. In particular, if $A \subset M$ is a Borel subset we set

$$\int_{T \llcorner A} \psi = (T \llcorner A)(\psi) := \int_T (\psi \cdot 1_A),$$

where 1_A denotes the characteristic function of A .

2.1.3. Manifolds and bundles. If M is a smooth manifold then we denote by S^*M its cosphere bundle. The elements of the total space of S^*M may be thought of either as rays lying in the fibers of the cotangent bundle with endpoint at 0, or else as oriented hyperplanes through the origin within the tangent spaces $T_x M$. For convenience we will sometimes make use of an arbitrarily chosen C^1 1-homogeneous length function $\ell : T^*M \rightarrow [0, \infty)$, positive off of the zero section (for example, one induced by a Riemannian metric on M). In this case we may also think of S^*M as the subspace $\ell^{-1}(1) \subset T^*M$. We put

$$\nu : T^*M \setminus (\text{zero-section}) \rightarrow S^*M$$

for the canonical projection. If M is Riemannian then $SM \subset TM$ is the bundle of unit tangent vectors. Abusing notation, we put π for the projection of any of the bundles TM , T^*M , SM , S^*M to M , and $\nu : TM \setminus (\text{zero-section}) \rightarrow SM$ for the normalization map.

We will frequently consider the case of a homogeneous space $M = G/G_o$, where G is a finite dimensional Lie group and G_o is the stabilizer of a base point $o \in M$.

We orient G, G_o, M so that whenever $M \supset U \ni x \mapsto \omega_x \in G$ is a smooth local section, $\omega_x o = x$, the map

$$(2.4) \quad U \times G_o \ni (x, \bar{\gamma}) \mapsto \bar{\gamma} \omega_x^{-1}$$

is an orientation-preserving diffeomorphism onto the corresponding open subset of G — there exist four different choices of such a system of orientations, but the distinctions among them will be immaterial. We denote by \mathcal{F} the bundle over $M \times M$ with fiber G_o and total space

$$\mathbf{F} := M \times M \times_{\mathcal{F}} G_o := \{(x, y, \gamma) \in M \times M \times G : \gamma y = x\}.$$

We orient these spaces consistently with the local product structure, i.e. if ω_x, ω_y are local sections as above defined on open subsets U, V then

$$(2.5) \quad (x, y, \bar{\gamma}) \mapsto (x, y, \omega_x \bar{\gamma} \omega_y^{-1})$$

yields an orientation-preserving local diffeomorphism $U \times V \times G_o \rightarrow \mathbf{F}$. One may easily check that with these conventions the diffeomorphism

$$(2.6) \quad M \times G \rightarrow M \times M \times_{\mathcal{F}} G_o, \quad (x, \gamma) \mapsto (x, \gamma^{-1}x, \gamma)$$

also preserves orientations. Abusing notation, we will use the same notation to denote pullbacks of \mathcal{F} by maps into $M \times M$, e.g.

$$TM \times TM \times_{\mathcal{F}} G_o := \{(\xi, \eta, \gamma) \in TM \times TM \times G : \gamma \pi(\eta) = \pi(\xi)\},$$

or, if S, T are currents living in TM , then $S \times T \times_{\mathcal{F}} G_o$ is the current given in an orientation-preserving local trivialization as the Cartesian product.

We will put Γ for the projection of $\mathbf{F} \subset M \times M \times G$ to the third factor, and X, Y the projections to the respective M factors. We will abuse notation by using the same symbols also to denote the maps on pullbacks of \mathcal{F} obtained by precomposing with the associated maps into \mathbf{F} .

By the orientation conventions (2.1) and (2.6),

$$(2.7) \quad X_* \langle \mathbf{F}, \Gamma, \gamma \rangle = (-1)^{d \dim G} M.$$

Define the involutions

$$\begin{aligned} \iota : \mathbf{F} &\rightarrow \mathbf{F}, (x, y, \gamma) \mapsto (y, x, \gamma^{-1}), \\ I : G &\rightarrow G, \gamma \mapsto \gamma^{-1}, \end{aligned}$$

both of parity $(-1)^{\dim G} = (-1)^{d + \dim G_o}$. Conclusions (6) and (7) of [9], Theorem 4.3.2 now yield

$$\begin{aligned} (2.8) \quad Y_* \langle \mathbf{F}, \Gamma, \gamma \rangle &= (-1)^{\dim G} Y_* \langle \mathbf{F}, I \circ \Gamma, \gamma^{-1} \rangle \\ &= (-1)^{\dim G} Y_* \langle \mathbf{F}, \Gamma \circ \iota, \gamma^{-1} \rangle \\ &= (-1)^{\dim G} Y_* \iota_* \langle \iota_* \mathbf{F}, \Gamma, \gamma^{-1} \rangle \\ &= (Y \circ \iota)_* \langle \mathbf{F}, \Gamma, \gamma^{-1} \rangle \\ &= X_* \langle \mathbf{F}, \Gamma, \gamma^{-1} \rangle \\ &= (-1)^{d \dim G} M. \end{aligned}$$

If M is Riemannian and G acts on M by isometries then we may endow G with an invariant volume form $d\gamma$ compatible with the given orientation, such that the corresponding volume measure on G projects to the positively oriented Riemannian volume form $d\text{Vol}_M$ of M (i.e. $d\gamma$ is the product of the Riemannian volume of M with the invariant probability volume form on the compact subgroup G_o). Let

$\pi_{G_o*} : \Omega^*(\mathbf{F}) \rightarrow \Omega^*(M \times M)$ denote fiber integration over G_o . Since the maps (2.4), (2.5) preserve orientation, we observe that

$$(2.9) \quad \pi_{G_o*}(\Gamma^* d\gamma) \equiv Y^*(d \text{Vol}_M) \mod X^* \Omega^*(M),$$

$$(2.10) \quad \pi_{G_o*}(\Gamma^* d\gamma) \equiv (-1)^d X^*(d \text{Vol}_M) \mod Y^* \Omega^*(M).$$

2.1.4. *Convexity.* By a *convex body* we understand a non-empty, compact and convex subset of \mathbb{R}^d . If K is a convex body and $n \in S^{d-1}$ a unit vector, the *support function* of K at n is

$$h_K(n) = \sup\{x \cdot n : x \in K\}.$$

For $t > 0$ we denote by

$$C(K, n, t) := \{x \in K : x \cdot n \geq h_K(n) - t\}$$

the *cap* of K of direction n and width t . If $x \in \partial K$ we write

$$\text{nor}(K, x)$$

for the set of all unit outer normal vectors to K at x (these are vectors from the dual cone to the tangent cone of K at x). The *width* of K is defined as

$$\text{width } K = \inf\{h_K(n) + h_K(-n) : n \in S^{d-1}\}.$$

The symbol Δ will denote the difference operator on sets:

$$\Delta A := A - A = \{a - b : a, b \in A\}.$$

2.2. Minkowski content.

Definition 2.1. The *m-dimensional upper Minkowski content* of $S \subset \mathbb{R}^d$ is

$$\mathcal{M}^{*m}(S) = \limsup_{\varepsilon \downarrow 0} (2\varepsilon)^{m-d} \text{Vol}(S_\varepsilon),$$

where S_ε is the set of points in M lying within distance ε of S . If $\mathcal{M}^{*m}(S) < \infty$ then we say that S has **finite m-content**.

For S as above and $\varepsilon > 0$, we define the *ε -covering number* of S

$$\#(S, \varepsilon) = \min \left\{ k : S \subset \bigcup_{i=1}^k B(x_i, \varepsilon) \text{ for some } x_1, \dots, x_k \in \mathbb{R}^d \right\}.$$

Lemma 2.2.

(1) *S has finite m-content iff*

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-m} \#(S, \varepsilon) < \infty.$$

(2) *If S has σ -finite m-content then S has Hausdorff dimension $\leq m$.*

(3) *If S has finite m-content and T has finite n-content, then $S \times T$ has finite $(m+n)$ -content.*

Proof. (1) follows from the inequalities

$$P(S, \varepsilon) \omega_d \varepsilon^d \leq \text{Vol}(S_\varepsilon) \leq \#(S, 2\varepsilon) \omega_d (2\varepsilon)^d,$$

where $P(S, \varepsilon)$ is the maximal number of disjoint ε -balls with centres in S (ε -packing number of S), and from

$$\#(S, 2\varepsilon) \leq P(S, \varepsilon) \leq \#(S, \varepsilon/2),$$

see [20, §5.3-5.5] for details. Assertions (2) and (3) follow at once from (1). \square

2.3. Lipschitz and DC functions. If f is a locally Lipschitz function defined on a d -dimensional C^1 manifold M , we denote by $\partial f(x)$ its **Clarke differential** [7] at $x \in M$. To be explicit, we take $\partial f(x) \subset T_x^*M$ to be the convex hull of the set of all $\xi \in T_x^*M$ with the following property: there exists a sequence $x_1, x_2, \dots \rightarrow x$, such that f is differentiable at each x_i , and $\lim_{i \rightarrow \infty} df(x_i) = \xi$. Since by Rademacher's theorem such f is differentiable a.e., this defines a nonempty compact convex subset of T_x^*M . If ψ is C^1 then the chain rule

$$\partial(f \circ \psi)(x) \subset \psi^*(\partial f(\psi(x)))$$

holds (cf. [7] for this and other basic relations regarding the Clarke differential).

Lemma 2.3. *If x is a local extremum of f then $0 \in \partial f(x)$. \square*

Lemma 2.4 ([7], Proposition 2.3.3). *If $f, g : M \rightarrow \mathbb{R}$ are locally Lipschitz functions then*

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x),$$

where $+$ on the right hand side denotes Minkowski sum.

Definition 2.5. Put

$$\text{graph}(\partial f) := \{\xi \in T^*M : \xi \in \partial f(\pi(\xi))\}.$$

Clearly $\text{graph}(\partial f)$ is a closed subset of T^*M .

A number $c \in \mathbb{R}$ is a **weakly regular value** of f if

$$(2.11) \quad \text{clos}(\text{graph}(\partial f) \cap \pi^{-1}f^{-1}((c, \infty))) \cap \pi^{-1}f^{-1}(c) \cap (\text{zero-section}) = \emptyset.$$

This condition is equivalent to each of the following statements:

- (1) Whenever $M \ni x_1, x_2, \dots \rightarrow x_0$, with $f(x_i) > f(x_0) = c$, and $\xi_i \in \partial f(x_i)$, then $\xi_i \not\rightarrow 0$.
- (2) Let $\ell : T^*M \rightarrow [0, \infty)$ be a length function as above. Then for any $K \subset\subset M$ there exists $\varepsilon > 0$ such that

$$x \in K, v \in \partial f(x), c < f(x) < c + \varepsilon \implies \ell(v) \geq \varepsilon.$$

Definition 2.6. A function f defined on an open set $U \subset \mathbb{R}^d$ is **DC** if for every $x \in U$ there is some convex neighborhood $V \subset U$ of x , and convex functions $g, h : V \rightarrow \mathbb{R}$, such that $f = g - h$ on V . The class of all such functions is denoted $\text{DC}(U)$.

Obviously every DC function is locally Lipschitz. The class of DC functions enjoys many remarkable properties, prominently the following classical result of Hartman.

Theorem 2.7 ([14]). *Let $U \subset \mathbb{R}^n, V \subset \mathbb{R}^d$ be open. Suppose $\psi = (\psi_1, \dots, \psi_n) : V \rightarrow U$, where the $\psi_i \in \text{DC}(V)$, and $f \in \text{DC}(U)$. Then $f \circ \psi \in \text{DC}(V)$.*

Thus if M is a $C^{1,1}$ manifold we may define $\text{DC}(M)$ to be the space of all functions $f : M \rightarrow \mathbb{R}$ with the property that $f \circ \psi^{-1} \in \text{DC}(U)$ whenever (ψ, U) is a $C^{1,1}$ coordinate patch for M .

Corollary 2.8. *If $f, g \in \text{DC}(M)$ then $f + g, f \vee g := \max(f, g) \in \text{DC}(M)$.*

2.4. WDC sets. The definition of these objects is motivated by the following. Recall that a function f defined on an open subset of \mathbb{R}^d is **semiconvex** if it may be expressed locally as the sum of a convex function and a smooth function. It is clear that any semiconvex function, and in particular any $C^{1,1}$ function, is DC.

Theorem 2.9 (Kleinjohann [18], Bangert [2]). *A set $A \subset U \subset \mathbb{R}^d$ has locally positive reach iff $A = f^{-1}(-\infty, 0]$, where $f : U \rightarrow \mathbb{R}$ is a semiconvex function and 0 is a weakly regular value of f .*

Definition 2.10. Let M be a C^2 manifold. A subset $A \subset M$ is a **WDC subset** of M (or simply a **WDC set**) if $A = f^{-1}(-\infty, c]$ for some $f \in \text{DC}(M)$ and some weakly regular value c of f .

If $c = 0$ and $f \geq 0$ then f is a **DC aura** (or simply an **aura**) for A .

Remark. This terminology is different from that of [12], in which the function f would be referred to as a *nondegenerate aura*. The point there was that the weak regularity condition may be removed if the function involved is subanalytic. In the present paper, however, all auras will be nondegenerate.

Proposition 2.11. *Every WDC set admits a DC aura.*

Proof. Given A, f, c as above, Corollary 2.8 implies that $(f - c) \vee 0$ is a DC aura for A . \square

Definition 2.12. Let M be a C^2 manifold and f an aura for a WDC set $A \subset M$. Given $\varepsilon > 0$ we denote

$$\text{nor}_\varepsilon f := \nu(\{v : \ell(v) \geq \varepsilon, \text{ and } v \in \partial f(x) \text{ for some } x \in \text{bdry } A\}) \subset S^*M.$$

Since the graph of the Clarke differential ∂f is closed, it follows that $\text{nor}_\varepsilon(f)$ is a closed subset of S^*M for every $\varepsilon > 0$.

2.5. Monge-Ampère functions. Let M be an oriented C^2 manifold of dimension d . The cotangent bundle T^*M carries a natural **canonical 1-form** $\alpha \in \Omega^1(T^*M)$ given by

$$\langle \alpha_\xi, \tau \rangle := \langle \xi, \pi_* \tau \rangle, \quad \xi \in T^*M, \tau \in T_\xi T^*M.$$

The exterior derivative $\omega := d\alpha \in \Omega^2(T^*M)$ is the standard symplectic form of T^*M .

We recall that $f \in W_{loc}^{1,1}(M)$ is said to be **Monge-Ampère** (or **MA**) if there exists an integral current $\mathbb{D}(f) \in \mathbb{I}_d(T^*M)$ satisfying the axioms of [10, 16, 17], i.e.

- (1) $\partial \mathbb{D}(f) = 0$;
- (2) $\mathbb{D}(f)$ is Lagrangian, i.e. $\mathbb{D}(f) \llcorner \omega = 0$;
- (3) $\text{mass}(\mathbb{D}(f) \llcorner \pi^{-1}(K)) < \infty$ for every $K \subset\subset M$ (the mass may be computed with respect to the Sasaki metric corresponding to any C^2 Riemannian metric on M);
- (4) for any smooth volume form $d\text{Vol}_M \in \Omega^d(M)$ and every $g \in C_c^\infty(T^*M)$,

$$\int_{\mathbb{D}(f)} g \wedge \pi^*(d\text{Vol}_M) = \int_M g(x, df(x)) d\text{Vol}_M.$$

By Theorem 4.3.2(1) of [9], the condition (4) may be replaced by the equivalent condition

$$(4') \quad \langle \mathbb{D}(f), \pi, x \rangle = \delta_{(x, df(x))} \text{ for a.e. } x \in M.$$

As shown in the papers cited above, these axioms determine $\mathbb{D}(f)$ uniquely if it exists. We denote the class of all such f by $\text{MA}(M)$. Strictly speaking, the discussion there applies to the case where M is an open subset of \mathbb{R}^d ; starting from that formulation the class $\text{MA}(M)$ may also be defined in terms of local coordinates, in view of the following.

Lemma 2.13. *Let $U, V \subset \mathbb{R}^d$ be open, and $\psi : U \rightarrow V$ a C^2 diffeomorphism. Put $\tilde{\psi} = (\psi^{-1})^* : T^*U \rightarrow T^*V$ for the induced C^1 diffeomorphism of cotangent bundles. If $f \in \text{MA}(V)$ then $f \circ \psi \in \text{MA}(U)$, with*

$$\mathbb{D}(f \circ \psi) = \tilde{\psi}_* \mathbb{D}(f).$$

Proof. Using the fact that $\tilde{\psi}$ is a symplectomorphism, it is easy to confirm that the right hand side satisfies the axioms above with f replaced by $f \circ \psi$. \square

The starting point for the main constructions of this paper is the following.

Theorem 2.14 ([22]). *Every DC function is MA.* \square

We will also need the following fundamental fact.

Lemma 2.15 ([10]). *If f is a locally Lipschitz MA function then*

$$\text{spt } \mathbb{D}(f) \subset \text{graph } \partial f. \quad \square$$

2.5.1. Sums of MA functions on a homogeneous space. Although the class $\text{MA}(M)$ is not closed under addition, it is closed under addition in general position in a sense given in the next Theorem, a variant of Proposition 2.6 of [10], part I.

Let G be a Lie group and $M = G/G_o$ an oriented homogeneous space of G , where G_o is the stabilizer of the arbitrarily chosen base point $o \in M$. Abusing notation, we denote simply by γ the induced action of $\gamma \in G$ on T^*M by symplectomorphisms, i.e. pullback under the diffeomorphism $\gamma^{-1} : M \rightarrow M$:

$$(2.12) \quad \gamma\xi := (\gamma^{-1})^*\xi$$

Referring to the convention of Section 2.1.3, consider the smooth manifold

$$\tilde{\mathbf{F}} := T^*M \times T^*M \times_{\mathcal{F}} G_o.$$

We put $\Gamma : \tilde{\mathbf{F}} \rightarrow G$ for the map given by the restricted projection to G , and

$$\Sigma : \tilde{\mathbf{F}} \rightarrow T^*M, \quad \Sigma(\xi, \eta, \gamma) := \xi + \gamma\eta.$$

We put also $X, Y : \tilde{\mathbf{F}} \rightarrow M$ for the respective projections to the base spaces of the first and second factors.

Theorem 2.16. *Suppose the manifold M is an oriented homogeneous space of G , as above, and let $f, g \in \text{MA}(M)$. Then $h_\gamma := f + g \circ \gamma^{-1} \in \text{MA}(M)$ for a.e. $\gamma \in G$, with*

$$(2.13) \quad \mathbb{D}(h_\gamma) = (-1)^{d \dim G} \Sigma_* \langle \mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle.$$

Proof. We check that for a.e. $\gamma \in G$ the right hand side satisfies the axioms (1), (2), (3), (4') for MA functions, with f replaced by $f + g \circ \gamma^{-1}$.

Axiom (1) is immediate from (2.3).

To show (2), by [9], Theorem 4.3.2(1) it is enough to prove the following claim. Let $\Xi, H : \tilde{\mathbf{F}} \rightarrow T^*M$ denote the restrictions to $\tilde{\mathbf{F}} \subset T^*M \times T^*M \times G_o$ of the projections to the first and second factors, respectively. Then

$$(2.14) \quad \Sigma^*\omega \equiv 2\Xi^*\omega + 2H^*\omega \pmod{\Gamma^*\Omega^*(G)}.$$

In fact we prove the stronger claim that

$$(2.15) \quad \Sigma^*\alpha \equiv 2\Xi^*\alpha + 2H^*\alpha \pmod{\Gamma^*\Omega^*(G)},$$

from which (2.14) follows by taking the exterior derivative.

To prove (2.15), observe first that each tangent space

$$T_{(\xi, \eta, \gamma)} \tilde{\mathbf{F}} \subset T_\xi T^*M \oplus T_\eta T^*M \oplus T_\gamma G$$

and that the derivative Γ_* of Γ equals the projection to the last factor on the right. The kernel of this map is clearly

$$(2.16) \quad V := \{(\sigma, \tau, 0) : \sigma \in T_\xi T^*M, \tau \in T_\eta T^*M, \pi_*\sigma = \pi_*\gamma_*\tau \in T_{\pi(\xi)}M\}$$

and it is enough to show that

$$\Sigma^*\alpha|_V = 2\Xi^*\alpha|_V + 2H^*\alpha|_V.$$

But for $(\sigma, \tau, 0) \in V$

$$\begin{aligned}
\langle \Sigma^* \alpha, (\sigma, \tau, 0) \rangle &= \langle \alpha, \Sigma_*(\sigma, \tau, 0) \rangle \\
&= \langle \alpha, \sigma + \gamma_* \tau \rangle \\
&= \langle \xi + \gamma \eta, \pi_*(\sigma + \gamma_* \tau) \rangle \\
&= 2\langle \xi, \pi_* \sigma \rangle + 2\langle \gamma \eta, \gamma_* \pi_* \tau \rangle \quad \text{by (2.16)} \\
&= 2\langle \xi, \pi_* \sigma \rangle + 2\langle \eta, \pi_* \tau \rangle \quad \text{by (2.12)} \\
&= 2\langle \Xi^* \alpha + H^* \alpha, (\sigma, \tau, 0) \rangle
\end{aligned}$$

as claimed.

To prove (3), we wish to show that for a.e. $\gamma \in G$ and any $K \subset\subset M$

$$\text{mass}(\Sigma_*(\mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o, \Gamma, \gamma) \llcorner \pi^{-1}(K)) < \infty.$$

Since Σ is Lipschitz when restricted to the preimage of any compact subset of \mathbf{F} under the projection $\tilde{\mathbf{F}} \rightarrow \mathbf{F}$, the last relation on p. 370 of [9], together with Theorem 4.3.2(2), op. cit., imply that it is enough to show that

$$\text{mass}((\mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o) \llcorner (\pi \circ \Sigma)^{-1}(K) \cap \Gamma^{-1}(J)) < \infty$$

for $K \subset\subset M, J \subset\subset G$. But this last current is supported in $X^{-1}(K) \cap Y^{-1}(J^{-1}K)$, and hence the finiteness of the mass follows from the finiteness axiom (3) for the MA functions f, g .

Finally we prove (4'). By commutativity of pushforward and slicing it is enough to show that

$$(2.17) \quad (-1)^{d \dim G} \langle \langle \mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle, X, x \rangle = \delta_{(x, df(x))} \times \delta_{(\gamma^{-1}x, dg(\gamma^{-1}x))} \times \delta_{\gamma}$$

for a.e. $(x, \gamma) \in M \times G$.

Clearly we may cover $M \times M$ by open sets $U \times V$ such that there exists a smooth local section $\omega : U \cup V \rightarrow G$, i.e. $\omega_z \cdot o = z$ for $z \in U \cup V$. For such U, V , consider the diagram

$$\begin{array}{ccc}
T^*M \times T^*M \times G_o \supset \pi^{-1}(U) \times \pi^{-1}(V) \times G_o & \xrightarrow{\tilde{\Phi}} & (X, Y)^{-1}(U \times V) \subset \tilde{\mathbf{F}} \\
\downarrow (X, Y, \bar{\Gamma}) & & \downarrow (X, \Gamma) \\
U \times V \times G_o & \xrightarrow{\Phi} & U \times G
\end{array}$$

where, abbreviating $x := \pi(\xi), y := \pi(\eta)$ for $\xi, \eta \in T^*M$, the vertical map on the left is the projection, the vertical map on the right is

$$(\xi, \eta, \gamma) \mapsto (x, \gamma),$$

and

$$\begin{aligned}
\tilde{\Phi}(\xi, \eta, \bar{\gamma}) &:= (\xi, \eta, \omega_x \bar{\gamma} \omega_y^{-1}) \\
\Phi(x, y, \bar{\gamma}) &:= (x, \omega_x \bar{\gamma} \omega_y^{-1}).
\end{aligned}$$

Thus $(X, Y) \circ \tilde{\Phi} = \Phi \circ (X, Y, \bar{\Gamma})$. By definition of the fiber product of currents (cf. Section 2.1.3),

$$(\mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o) \llcorner (X, Y)^{-1}(U \times V) = \tilde{\Phi}_*(\mathbb{D}(f|_U) \times \mathbb{D}(g|_V) \times G_o).$$

Note that Φ is a diffeomorphism onto its image, with inverse

$$(2.18) \quad \Phi^{-1}(x, \gamma) = (x, \gamma^{-1}x, \omega_x^{-1}\gamma\omega_y).$$

and preserves orientation by the convention (2.4). Thus for a.e. (x, γ) lying in this image, Theorem 4.3.2(6) and Theorem 4.3.5 of [9], together with the MA axiom

(4') for f, g , imply that

$$\begin{aligned}
(-1)^{d \dim G} \langle \mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle, X, x \rangle &= \\
&= (-1)^{d \dim G} \langle \mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o, (\Gamma, X), (\gamma, x) \rangle \\
&= \langle \mathbb{D}(f) \times \mathbb{D}(g) \times_{\mathcal{F}} G_o, (X, \Gamma), (x, \gamma) \rangle \\
&= \tilde{\Phi}_* \langle \mathbb{D}(f|_U) \times \mathbb{D}(g|_V) \times G_o, (X, \Gamma) \circ \tilde{\Phi}, (x, \gamma) \rangle \\
&= \tilde{\Phi}_* \langle \mathbb{D}(f|_U) \times \mathbb{D}(g|_V) \times G_o, \Phi \circ (X, Y, \bar{\Gamma}), (x, \gamma) \rangle \\
&= \tilde{\Phi}_* \langle \mathbb{D}(f|_U) \times \mathbb{D}(g|_V) \times G_o, (X, Y, \bar{\Gamma}), (x, \gamma^{-1}x, \omega_x^{-1}\gamma\omega_y) \rangle \\
&= \tilde{\Phi}_* (\delta_{(x, df(x))} \times \delta_{(\gamma^{-1}x, dg(\gamma^{-1}x))} \times \delta_{\omega_x^{-1}\gamma\omega_y}) \\
&= \delta_{(x, df(x))} \times \delta_{(\gamma^{-1}x, dg(\gamma^{-1}x))} \times \delta_{\gamma},
\end{aligned}$$

which is (2.17). \square

2.6. Conormal cycles for WDC sets. We give an abbreviated account of the DC case of the theory presented in [12]. It is remarkable that the ad hoc hypotheses needed to make [12] work out are always fulfilled here.

If $f \in \text{DC}(M)$ is an aura for a set $A = f^{-1}(0) \subset M$, we then make the following provisional definition, which will be superseded in view of Theorem 2.18 below:

$$(2.19) \quad N^*(f) := \nu_* \partial(\mathbb{D}(f) \llcorner \pi^{-1}A) = -\nu_* \partial(\mathbb{D}(f) \llcorner \pi^{-1}(M \setminus A)).$$

This object is clearly defined locally, in the sense that

$$(2.20) \quad N^*(f|_U) = N^*(f) \llcorner \pi^{-1}(U)$$

for any open subset $U \subset M$.

Theorem 2.17 ([12]). *Under the conditions above, $N^*(f)$ is an integral current of dimension $d-1$ in S^*M , with*

$$\partial N^*(f) = N^*(f) \llcorner \alpha = 0. \quad \square$$

We show that this current depends only on the underlying set A :

Theorem 2.18. *If $A \subset M$ is a WDC set, and f, g are DC auras for A , then*

$$N^*(f) = N^*(g). \quad \square$$

Definition 2.19. We define this common value to be $N^*(A)$.

Proof of Theorem 2.18. We show that $N^*(f), N^*(g)$ agree when restricted to any coordinate neighborhood (ψ, U) . Let $V := \psi(U) \subset \mathbb{R}^d$, and consider the DC functions $f \circ \psi^{-1}, g \circ \psi^{-1} : V \rightarrow \mathbb{R}$, which are both auras for $\psi(A \cap U) \subset V$, with

$$\begin{aligned}
\tilde{\psi}_*^{-1} N^*(f \circ \psi^{-1}) &= N^*(f) \llcorner \pi^{-1}(U), \\
\tilde{\psi}_*^{-1} N^*(g \circ \psi^{-1}) &= N^*(g) \llcorner \pi^{-1}(U).
\end{aligned}$$

Thus it is sufficient to show that every point $x \in V$ admits a neighborhood $W \subset V$ such that

$$N^*(f \circ \psi^{-1}) \llcorner \pi^{-1}(W) = N^*(g \circ \psi^{-1}) \llcorner \pi^{-1}(W).$$

Put $r > 0$ for the distance from x to the complement of V in \mathbb{R}^d , and let $h(y) := \max(0, |y| - \frac{r}{2})$ denote the standard aura for the closed disk $B := \bar{B}(0, \frac{r}{2})$. By the proof of Proposition 4.1 below, for a.e. euclidean motion γ the functions $(f \circ \psi^{-1}) + \gamma^* h, (g \circ \psi^{-1}) + \gamma^* h$ are both auras for $\psi(A \cap U) \cap \gamma B$. Clearly such γ may be chosen so that x lies in the interior of γB , and hence $\gamma B \subset V$. Therefore these functions may be extended to auras for $\psi(A \cap U) \cap \gamma B$, considered as a subset of \mathbb{R}^d .

It follows from Theorem 1.2 of [22] that

$$N^*((f \circ \psi^{-1}) + \gamma^* h) = N^*((g \circ \psi^{-1}) + \gamma^* h).$$

Since the restrictions of these currents to the interior of γB agree with those of $N^*(f \circ \psi^{-1})$, $N^*(g \circ \psi^{-1})$ respectively, this completes the proof. \square

Proposition 2.20. *If A is a compact WDC set with aura f , then*

$$\text{spt } N^*(A) \subset \text{nor}_\varepsilon f$$

for all sufficiently small $\varepsilon > 0$.

Proof. This follows at once from Lemma 2.15. \square

2.6.1. Conic cycles. We will need the following alternative construction of the conormal cycle, a restatement of Proposition 1.3 and equations (1.3d), (1.3g), (1.4c) of [12]. We identify S^*M with $\ell^{-1}(1)$ and define the maps

$$\begin{aligned} \nu : T^*M \setminus (\text{zero-section}) &\rightarrow S^*M, \quad \nu(\xi) := \frac{\xi}{\ell(\xi)}, \\ m : \mathbb{R} \times S^*M &\rightarrow T^*M, \quad m(t, \bar{\xi}) := t\xi, \\ m_t : T^*M &\rightarrow T^*M \quad m_t(\xi) := t\xi, t \in \mathbb{R}, \\ z &:= (\text{zero-section}) : M \rightarrow T^*M \end{aligned}$$

Proposition 2.21. *Let $f \in DC(M)$ be an aura for $A \subset\subset M$. Suppose U is a neighborhood of A and $r_0 > 0$ is small enough that $\ell(\xi) > r_0$ whenever $\pi(\xi) \in U \setminus A$. Then*

$$(2.21) \quad \mathbb{D}(f) \llcorner (\pi^{-1}(U) \cap \ell^{-1}[0, r_0]) = z_* A + m_*([0, r_0] \times N^*(A)),$$

$$(2.22) \quad N^*(A) = \nu_* \langle \mathbb{D}(f) \llcorner \pi^{-1}(U), \ell, r \rangle \quad \text{for all } r \in (0, r_0),$$

$$(2.23)$$

$$\vec{N}^*(A) := z_* A + m_*([0, \infty) \times N^*(A)) = \lim_{t \rightarrow \infty} m_{t*}(\mathbb{D}(f) \llcorner \pi^{-1}(U)),$$

$$(2.24) \quad N^*(A) = \langle \vec{N}^*(A), \ell, 1 \rangle. \quad \square$$

Remark. Note that the compactness of A ensures that such U, r_0 exist. The limit in (2.23) exists in a particularly strong sense: for any $C > 0$, there exists t_0 such that for $t > t_0$ the restrictions to $\ell^{-1}[0, C)$ of the left hand side and the expression under the limit agree.

3. THE SIZE OF THE CLARKE DIFFERENTIAL OF A DC FUNCTION

The main result of this section follows. It is a sharpened version of Proposition 7.1 of [22].

Theorem 3.1. *Let f be a DC aura on a Riemannian manifold M of dimension d and $\varepsilon > 0$. Then $\text{nor}_\varepsilon f$ has locally finite $(d-1)$ -content.*

Using local coordinates it is clearly enough to prove the theorem in the Euclidean case, so we shall assume throughout this section that $M = \mathbb{R}^d$. We follow the scheme of Pavlica and Zajíček [21], relating a fundamental result about the boundary structure of convex bodies (due to Ewald, Larman and Rogers [6]) to the set of tangents to the graph of a DC function. The classical result of [6] is enough to establish the conclusion of Theorem 2.18 in case the ambient manifold $M = \mathbb{R}^d$ (viz. Proposition 7.1 of [22]). However, for our present purposes we need the following more detailed version, where we use the notation of Section 2.1.4:

Lemma 3.2. *Let A, B be compact convex subsets of \mathbb{R}^d . Then the set*

$$\Sigma_{A,B} := \{(x - y, \xi) \in \mathbb{R}^d \times S^{d-1} : \xi \in \text{nor}(A, x) \cap \text{nor}(B, y)\}$$

has finite $(d - 1)$ -content.

Following the approach of [21], we will combine this Lemma with a duality between the space of tangents to graphs of DC functions $f - g$ and the structure of $\Sigma_{A,B}$, with A, B taken as the epigraphs of the convex conjugates (Legendre transforms) of f, g . This will establish a natural overestimate of the size of the support of the differential cycle of a DC function; exploiting the enhancements introduced in Lemma 3.2, this sharpens Proposition 7.1 of [21] (see Lemma 3.5). A simple argument then shows that this yields the desired estimate of the Minkowski content of $\text{nor}_\varepsilon f$ for DC auras f .

Remark. The corresponding lemma of [6] was weaker in two senses: it applies only to the projection of $\Sigma_{A,B}$ to the first (\mathbb{R}^d) factor, and it concludes only that this projection has locally finite $(d - 1)$ -dimensional Hausdorff measure. As an aside from our main application, we present at the end of this section a proof of a sharper version of the main theorem of [6] incorporating both of our improvements.

3.1. Normals to pairs of convex sets. In this section we prove Lemma 3.2.

Lemma 3.3. *Let $d \in \mathbb{N}$. There is a constant C_d such that for every $r > 0$ the following statement holds: If $K \subset \mathbb{R}^d$ is a convex body and $r \leq \text{width } K$ then K can be covered by M balls of radius r such that $Mr^d \leq C_d \cdot \text{Vol}_d(K)$.*

Proof. The lemma is a simple application of [6, Lemma 7]. Let \tilde{C}_d be a constant such that every ball $B \subset \mathbb{R}^d$ of a radius ρ can be covered by M balls of radius $r < \rho$ such that

$$(3.1) \quad Mr^d \leq \tilde{C}_d \cdot \omega_d \cdot \rho^d.$$

Fix a convex body $K \subset \mathbb{R}^d$ and $r \leq \text{width } K$. Using [6, Lemma 7] we can cover K by balls B_1, \dots, B_{M_1} of diameter $\delta := \sqrt{d} \cdot \text{width } K$ with

$$(3.2) \quad M_1 \cdot (\text{width } K)^d \leq 2^d \sqrt{d} \cdot d! \cdot \text{Vol}_d(K).$$

Using (3.1) we can cover every ball B_i by balls $B_1^i, \dots, B_{M_2}^i$ of diameter r such that

$$(3.3) \quad M_2 \cdot r^d \leq \tilde{C}_d \cdot \delta^d \cdot \omega_d.$$

Multiplying (3.2) and (3.3) we obtain

$$M_1 \cdot M_2 \cdot r^d (\text{width } K)^d \leq 2^d \sqrt{d} \cdot d! \tilde{C}_d (\sqrt{d})^d (\text{width } K)^d \omega_d \cdot \text{Vol}_d(K).$$

This means that K can be covered by $M := M_1 \cdot M_2$ balls of the form B_j^i , $i = 1, \dots, M_1$, $j = 1, \dots, M_2$, all of diameter r , with

$$Mr^d \leq C_d \cdot \text{Vol}_d(K),$$

where $C_d := 2^d d! \tilde{C}_d (\sqrt{d})^{d+1} \omega_d$. □

Lemma 3.4. *Let $K \subset \mathbb{R}^d$ be a convex body. Then for $\tilde{K} := K + B(0, 1)$, $0 < t < 1$, and $\nu \in S^{d-1}$, the spherical diameter of the set*

$$N_{\nu,t}(\tilde{K}) := \bigcup_{x \in C(\tilde{K}, \nu, t)} \text{nor}(\tilde{K}, x)$$

is smaller than $2\sqrt{3t}$.

Proof. Let $\nu \in \text{nor}(\tilde{K}, x_0)$, and $n \in \text{nor}(\tilde{K}, x)$, $x \in C(\tilde{K}, \nu, t)$. Then

$$x_0 \cdot \nu - t \leq x \cdot \nu \leq x_0 \cdot \nu.$$

Furthermore $x - n \in K$, and therefore $x - n + \nu \in \tilde{K}$. In particular

$$x \cdot \nu + (1 - n \cdot \nu) = (x - n + \nu) \cdot \nu \leq x_0 \cdot \nu,$$

so

$$(3.4) \quad \cos \theta = n \cdot \nu \geq 1 - t$$

where θ denotes the spherical distance between n, ν . Since $\cos \sqrt{3t} < 1 - t$ for $0 < t < 1$, we find that $\theta < \sqrt{3t}$, and the conclusion follows. \square

We are now ready to prove Lemma 3.2. The proof is an enhancement of a part of the proof of Theorem 1 from [6].

Proof of Lemma 3.2. First note that the set $\Sigma_{A,B} \subset \Sigma_{\tilde{A},\tilde{B}}$ with $\tilde{A} := A + B(0, 1)$ and $\tilde{B} := B + B(0, 1)$ (in fact, equality holds, but we will not need it in our proof). Indeed, if we choose $(x - y, p) \in \Sigma_{A,B}$ with $p \in \text{nor}(A, x) \cap \text{nor}(B, y)$, then

$$p \in (\text{nor}(\tilde{A}, x + p) \cap \text{nor}(\tilde{B}, y + p))$$

and, of course, $(x + p) - (y + p) = x - y$. Define $K := \tilde{A} + \tilde{B}$.

By [6, Lemma 5], there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ there are $n_1, \dots, n_M \in S^{d-1}$, $t_1, \dots, t_M > 0$ and a constant $D' = D'(d, A, B)$ such that

$$(3.5) \quad \partial K \subset \bigcup_{i=1}^M C(K, n_i, t_i), \quad \sum_{i=1}^M \text{Vol}_d(C(K, n_i, t_i)) \leq D' \varepsilon,$$

and

$$(3.6) \quad \text{width } C(K, n_i, t_i) \in [2\varepsilon, 36d\varepsilon]$$

for every i . We shall assume that $\varepsilon_0 < \frac{1}{72d}$, ensuring that these widths are all $\leq \frac{1}{2}$. Since

$$(3.7) \quad K = (A + B) + B(0, 2)$$

and by the fact that $\text{width } C(K, n_i, t_i) \leq \frac{1}{2}$ for every i , one can see that $\text{width } C(K, n_i, t_i) = t_i$ for every i . Hence (3.6) is equivalent to

$$(3.8) \quad t_1, \dots, t_M \in [2\varepsilon, 36d\varepsilon].$$

Select points $x_i \in C(A, n_i, t_i)$, $y_i \in C(B, n_i, t_i)$, and denote $z_i := x_i - y_i$. We show that

$$(3.9) \quad \Sigma_{A,B} \subset \bigcup_{i=1}^M \left[(z_i + \Delta C(K, n_i, 2t_i)) \times (N_{n_i, t_i}(\tilde{A}) \cap N_{n_i, t_i}(\tilde{B})) \right].$$

Let $(u, p) \in \Sigma_{A,B} \subset \Sigma_{\tilde{A},\tilde{B}}$, i.e., $u = x - y$ with $x \in \tilde{A}$, $y \in \tilde{B}$ and $p \in \text{nor}(\tilde{A}, x) \cap \text{nor}(\tilde{B}, y)$. Then there exists $i \leq M$ such that

$$(3.10) \quad x + y \in C(K, n_i, t_i).$$

We claim that

$$(3.11) \quad x \in C(\tilde{A}, n_i, t_i), \quad y \in C(\tilde{B}, n_i, t_i).$$

If not, then we may assume for definiteness that $x \cdot n_i < h_{\tilde{A}}(n_i) - t_i$, in which case additivity of support functions gives

$$(x + y) \cdot n_i < h_{\tilde{A}}(n_i) - t_i + h_{\tilde{B}}(n_i) = h_K(n_i) - t_i,$$

contradicting (3.10). Since (3.11) implies that $p \in N_{n_i, t_i}(\tilde{A}) \cap N_{n_i, t_i}(\tilde{B})$, in order to prove (3.9) it thus remains only to show that $u - z_i \in \Delta C(K, n_i, 2t_i)$. To this end we note that (3.11) implies that

$$x - y \in C(\tilde{A}, n_i, t_i) - C(\tilde{B}, n_i, t_i)$$

and so

$$\begin{aligned} u - z_i = (x - y) - (x_i - y_i) &\in \Delta(C(\tilde{A}, n_i, t_i) - C(\tilde{B}, n_i, t_i)) \\ &= \Delta(C(\tilde{A}, n_i, t_i) + C(\tilde{B}, n_i, t_i)) \\ &= \Delta C(K, n_i, 2t_i), \end{aligned}$$

as claimed. Here we have used that

$$\Delta(U - V) = \Delta(U + V), \quad U, V \subset \mathbb{R}^d,$$

and

$$C(K_1, n, t_1) + C(K_2, n, t_2) = C(K_1 + K_2, n, t_1 + t_2),$$

K_1, K_2 convex bodies, $n \in S^{d-1}$, $t_1, t_2 > 0$.

Given $1 \leq i \leq M$, put $H_i := \{x : x \cdot n_i = h_K(n_i) - 2t_i\}$ for the hyperplane containing the base P_i of the cap $C(K, n_i, 2t_i)$. In fact this base may be described either as the intersection $H_i \cap K$, or alternatively as the projection of the cap onto H_i , i.e.

$$\Pi_{H_i}(C(K, n_i, 2t_i)) = K \cap H_i = C(K, n_i, 2t_i) \cap H_i =: P_i.$$

If not, then there exists an interior point $x \in C(K, n_i, 2t_i)$ such that the segment $[x, \Pi_{H_i}(x)]$ intersects the boundary of K at some point $y \in \partial K$, and there exists a unit outer normal vector $v \in \text{nor}(K, y)$ with $v \cdot n_i \leq 0$. But since by (3.7) the unit ball is a Minkowski summand of K , the relation (3.4) implies that $n_i \cdot v \geq 1 - 2t_i > 0$, which is a contradiction.

By linearity, the same is true of the corresponding difference sets, i.e.

$$\Pi_{\bar{H}_i}(\Delta C(K, n_i, 2t_i)) = \Delta P^i = \Delta C(K, n_i, 2t_i) \cap \bar{H}_i,$$

where $\bar{H}_i := \{x : x \cdot n_i = 0\}$. Clearly $\Delta C(K, n_i, 2t_i)$ contains the union of two disjoint antipodal cones with common base ΔP^i and heights $2t_i$, and therefore

$$\text{Vol}_d(\Delta C(K, n_i, 2t_i)) \geq \frac{4t_i}{d} \text{Vol}_{d-1} \Delta P^i \geq \frac{8\varepsilon}{d} \text{Vol}_{d-1}(\Delta P^i).$$

Using (3.5) we conclude that

$$(3.12) \quad \sum_{i=1}^M \text{Vol}_{d-1}(\Delta P^i) \leq D := \frac{D'd}{8}.$$

Using (3.7) it is easy to see that P_i , and hence ΔP^i also, includes a ball of radius $\sqrt{t_i}$. Thus Lemma 3.3 implies that ΔP^i may be covered by Q_i balls of diameter $\sqrt{t_i} \geq \sqrt{2\varepsilon}$ such that

$$(3.13) \quad Q_i 2^{\frac{d-1}{2}} \varepsilon^{\frac{d-1}{2}} \leq Q_i t_i^{\frac{d-1}{2}} \leq C_{d-1} \cdot \text{Vol}_{d-1}(\Delta P^i).$$

Since $\Delta C(K, n_i, 2t_i) \subset \Delta P^i + [-2t_i, 2t_i]n_i$ for every i , we conclude that every set $\Delta C(K, n_i, 2t_i)$ can be covered by Q_i balls of diameter $5\sqrt{t_i}$. Denote these balls by $B_1^i, \dots, B_{Q_i}^i$.

By Lemma 3.4 we have a fortiori

$$\text{diam}(N_{n_i, t_i}(\tilde{A}) \cap N_{n_i, t_i}(\tilde{B})) \leq 2\sqrt{3t_i}$$

for every i . Therefore by (3.9) the set $\Sigma_{\tilde{A}, \tilde{B}}$ can be covered by $\sum_i^M Q_i$ sets of the form

$$(z_i + B_j^i) \times (N_{n_i, t_i}(\tilde{A}) \cap N_{n_i, t_i}(\tilde{B})) \subset \mathbb{R}^d \times S^{d-1}, \quad i = 1, \dots, M, \quad j = 1, \dots, Q_i,$$

each of which has diameter at most $7\sqrt{t_i} \leq 42\sqrt{d}\sqrt{\varepsilon}$. Since by (3.12) and (3.13)

$$(\sqrt{\varepsilon})^{d-1} \sum_i^M Q_i \leq 2^{-\frac{d-1}{2}} C_{d-1} D,$$

where the right hand side is independent of ε , Lemma 2.2 (1) completes the proof. \square

3.2. The Minkowski dimension of $\text{nor}_\varepsilon f$.

Lemma 3.5. *If $f \in DC(\mathbb{R}^d)$ then*

$$\text{graph } \partial f \subset T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d$$

has locally finite d -content.

Proof. This follows by an adaptation of the duality argument by Pavlica and Zajíček in [21, Proposition 4.2]. Let $f = g - h$ for convex functions g, h . We shall show that

$$E_K(g, h) := \{(x, u - v) : x \in K, u \in \partial g(x), v \in \partial h(x)\}$$

has finite d -content for every $K \subset \subset \mathbb{R}^d$, which will imply the assertion by Lemma 2.4.

Suppose that both g and h are Lipschitz with a constant L on an open neighbourhood U of K and let $g^*(x) := \sup_\xi (x \cdot \xi - g(\xi))$, $h^*(x)$ be the respective conjugate functions to g, h . We may assume that both g^* and h^* are finite everywhere; this is equivalent to the assumption that $\bigcup_x \partial g(x) = \bigcup_x \partial h(x) = \mathbb{R}^d$ (cf. [21, Lemma 2.4]), which in turn may be assured by altering g, h outside of U if necessary. Thus ([23, Proposition 11.3])

$$(u \in \partial g(x) \ \& \ x \in K) \implies (x \in \partial g^*(u) \ \& \ u \in B(0, L))$$

and similarly for h , so that

$$E_K(g, h) \subset \{(x, u - v) : u, v \in B(0, L), x \in \partial g^*(u) \cap \partial h^*(v)\} =: \tilde{E}_L(g, h).$$

Let $A \subset \text{epi } g^*, B \subset \text{epi } h^*$ be compact convex subsets of \mathbb{R}^{d+1} whose boundaries include the graphs of $g^*|_{B(0, L)}, h^*|_{B(0, L)}$ respectively. Since

$$\text{nor}(\text{epi } g^*, (u, g^*(u))) = \left\{ \frac{(x, -1)}{\sqrt{1 + |x|^2}} : x \in \partial g^*(u) \right\},$$

and analogously for h^* , we find that the set $\Sigma_{A, B}$ from Lemma 3.2 includes the set

$$\tilde{\Sigma}_{A, B} = \left\{ \left(u - v, g^*(u) - h^*(v), \frac{(x, -1)}{\sqrt{1 + |x|^2}} \right) : u, v \in B(0, L), x \in \partial g^*(u) \cap \partial h^*(v) \right\},$$

which therefore has finite d -content. Since $\tilde{E}_L(g, h)$ is a subset of the image of $\tilde{\Sigma}_{A, B}$ under the mapping

$$(a, t, b, s) \mapsto \left(-\frac{b}{s}, a \right),$$

which is Lipschitz on the set where $|b|^2 + s^2 = 1, -\frac{b}{s} \in K$, this completes the proof. \square

Proof of Theorem 3.1. Let f be a DC aura in \mathbb{R}^d . We claim that the graph of ∂f includes a subset that is locally biLipschitz equivalent to the Cartesian product of $\text{nor}_\varepsilon f$ with an interval. With Lemma 3.5 this implies the desired conclusion.

By the definition of $\text{nor}_\varepsilon f$, if $\xi = (x, u) \in \text{nor}_\varepsilon f$ then $tu \in \partial f(x)$ for some $t \geq \varepsilon$. By Lemma 2.3, $0 \in \partial f(x)$ whenever $f(x) = 0$, so by the convexity of $\partial f(x)$ it follows that the whole segment $[0, \varepsilon u] \subset \partial f(x)$. Thus the map

$$((x, u), t) \mapsto (x, tu)$$

yields a biLipschitz embedding of $\text{nor}_\varepsilon f \times [0, \varepsilon]$ into $\text{graph } \partial f$. Since the latter set has locally finite d -content the conclusion of Theorem 3.1 follows. \square

Proof of Theorem A. Clearly it is sufficient to prove the statement in the case where the convex set K is compact. Under this assumption, let H, H' be distinct parallel hyperplanes that intersect K . Let $T_K^{H, H'}$ be the subset of T_K from Theorem A induced by boundary segments that intersect both H and H' . Since clearly T_K is the union of a countable family of subsets of such type, it will be sufficient to show that $T_K^{H, H'}$ has finite $(d-2)$ -content for a fixed pair H, H' .

Denote $A = K \cap H$ and $B = (K \cap H') - z$, where $z \perp H$ is the vector with $H' = H + z$; A, B are closed convex subsets of H . Observe that if $[x, \bar{y}]$ is a segment from the boundary of K that intersects both H and H' and with direction v and lying in a supporting hyperplane of outward normal direction w , then $(x - \bar{y} + z, \Pi_H w / |\Pi_H w|)$ belongs to one of the sets $\Sigma_{A, B}, \Sigma_{B, A} \subset H \times S^{d-2}$, where $S^{d-2} \subset H$ is the unit sphere of H . Now Lemma 3.2 yields that $\Sigma_{A, B}$ has finite $(d-2)$ -content. Inverting this mapping, we obtain that

$$(x - y, u) \mapsto \left(\frac{x - y + z}{|x - y + z|}, \frac{|z|^2 u - (u \cdot (x - y))z}{||z|^2 u - (u \cdot (x - y))z|} \right)$$

maps $\Sigma_{A, B} \cup \Sigma_{B, A}$ onto $T_K^{H, H'}$. Since both denominators in the formula above are bounded from below (by $|z|, |z|^2$, respectively), the mapping is Lipschitz. It follows that $T_K^{H, H'}$ has finite $(d-2)$ -content as well and the proof is finished. \square

4. PROOF OF THEOREM B

Throughout this section we take (M, G) to be a Riemannian isotropic space, i.e. M to be a Riemannian manifold and G a group of isometries of M that acts transitively on the tangent sphere bundle SM . We choose base points $\bar{o} \in SM, o \in M$ with $\pi(\bar{o}) = o$, and denote by $G_{\bar{o}}, G_o \subset G$ the respective stabilizers of these points. Thus we may identify $SM \simeq G/G_{\bar{o}}, M \simeq G/G_o$. It is clear that the space $\Omega^*(SM)^G$ of G -invariant differential forms on SM is isomorphic to the space $(\bigwedge^* T_{\bar{o}} SM)^{G_{\bar{o}}}$ of $G_{\bar{o}}$ -invariant elements of the exterior algebra of the tangent space to SM at \bar{o} . In particular, this space has finite dimension.

For $A \in \text{WDC}(M)$ we let $N(A)$ denote the **normal cycle** of A , i.e. the image of $N^*(A)$ under the diffeomorphism $S^*M \rightarrow SM$ induced by the Riemannian metric, and for a DC aura f and $\varepsilon > 0$ we define $\text{nor}_\varepsilon f \subset SM$ similarly. We also continue to use the notation $\mathbb{D}(f)$ for the “gradient cycle”, the image in TM of the differential cycle of $f \in \text{MA}(M)$.

4.1. Generic intersections. The first part of Theorem B is established in the following.

Proposition 4.1. *Let (M, G) be a Riemannian isotropic space and let f, g be DC auras for the compact sets $A, B \subset M$, respectively. Then there exists $C \subset G$ of measure zero such that $h_\gamma := f + g \circ \gamma^{-1}$ is an aura for $A \cap \gamma B$ whenever $\gamma \notin C$.*

More precisely, every $\gamma_0 \in G \setminus C$ admits a neighborhood $W \subset G \setminus C$ with the following property: there exist open sets $U \supset A, V \supset B$, and a constant $\varepsilon_0 > 0$, such that

$$\ell(\xi + \gamma\eta) > \varepsilon_0$$

whenever (abbreviating $x := \pi(\xi), y := \pi(\eta)$)

$$\gamma \in W,$$

$$\gamma y = x \in (U \cap \gamma V) \setminus (A \cap \gamma B),$$

$$\xi \in \partial f(x), \quad \eta \in \partial g(y).$$

Proof. By definition of weak regularity, we may find $\varepsilon > 0$ and neighborhoods $U' \supset A, V' \supset B$ such that $\ell(\xi), \ell(\eta) \geq \varepsilon$ whenever $\xi \in \text{graph}(\partial f) \cap \pi^{-1}(U' \setminus A), \eta \in \text{graph}(\partial g) \cap \pi^{-1}(V' \setminus B)$. Since A, B are compact, it follows that $\text{nor}_\varepsilon f, \text{nor}_\varepsilon g$ are compact as well. By Theorem 3.1 and Lemma 2.2 (3), the product $\text{nor}_\varepsilon(f) \times \text{snor}_\varepsilon(g)$ is compact, with finite $(2d - 2)$ -content, where $d = \dim M$.

Put $s : SM \rightarrow SM$ for the fiberwise antipodal map.

Consider

$$F' := \{(\xi, \eta, \gamma) \in SM \times SM \times G : \gamma\eta = -\xi\}.$$

The projection of F' to the first two factors is clearly a fiber bundle with fibers diffeomorphic to $G_{\bar{o}}$. Thus the preimage of $\text{nor}_\varepsilon f \times \text{nor}_\varepsilon g$ under the projection of F' is compact and has finite $(2d - 2 + \dim G_{\bar{o}})$ -content, so the projection C of this preimage to the third (G) factor has the same property. Since $\dim G = \dim SM + \dim G_{\bar{o}} = 2d - 1 + \dim G_{\bar{o}}$, it follows that C has measure zero in G .

Let $\gamma_0 \in G \setminus C$. We prove the more detailed statement of the second paragraph. By construction,

$$(4.1) \quad \text{nor}_\varepsilon f \cap s(\gamma_0 \text{nor}_\varepsilon g) = \emptyset.$$

If the conclusion is false then there exist $\xi_0, \eta_0 \in TM$ and sequences

$$\gamma_i \rightarrow \gamma_0, \quad \xi_i \rightarrow \xi_0, \quad \eta_i \rightarrow \eta_0$$

such that, putting $x_i := \pi(\xi_i), y_i := \pi(\eta_i)$:

$$\xi_i \in \partial f(x_i),$$

$$\eta_i \in \partial g(y_i),$$

$$U' \setminus A \ni x_i \rightarrow x_0 \in A \quad (\text{for definiteness}),$$

$$V' \ni y_i \rightarrow y_0 \in B,$$

$$\gamma_i y_i = x_i,$$

$$\xi_i + \gamma_i \eta_i \rightarrow 0.$$

Then $\ell(\xi_i) \geq \varepsilon$, so $\xi_0 \in \partial f(x_0) \cap \ell^{-1}[\varepsilon, \infty)$. Since by continuity $\xi_0 + \gamma_0 \eta_0 = 0$, it follows that $\eta_0 \in \partial g(y_0) \cap \ell^{-1}[\varepsilon, \infty)$, where $\gamma_0 y_0 = x_0$. This contradicts (4.1). \square

Remark. As a side point, a simpler version of this last proof yields the following, correcting an error in [22, Proposition 7.3] and [12, §2.2.3]. There it is stated that $f + g$ is an aura under a condition similar to but weaker than (4.2), with $\text{nor}_\varepsilon f$ replaced by a larger set. That statement may be true, but we do not know how to prove it.

Corollary 4.2. *Suppose $A, B \subset M$ are WDC sets, with auras f, g respectively. Suppose that for all sufficiently small $\varepsilon > 0$*

$$(4.2) \quad \text{nor}_\varepsilon f \cap s(\text{nor}_\varepsilon g) = \emptyset.$$

Then $A \cap B$ is a WDC set, with aura $f + g$.

4.2. The main diagram. Next we recall a classical construction of integral geometry, formalized current-theoretically in [11]. Consider the space

$$\mathbf{E} := \{(\xi, \eta, \zeta, \gamma) \in SM^3 \times G : \pi(\xi) = \pi(\zeta) = \gamma\pi(\eta)\},$$

to be thought of as the total space of a fiber bundle \mathcal{E} over $SM \times SM$, with fiber over $(\xi, \eta) \in SM \times SM$ given by

$$\mathbf{E}_{\xi, \eta} := \{(\zeta, \gamma) \in SM \times G : \pi(\zeta) = \pi(\xi) = \gamma\pi(\eta)\} \simeq S_o M \times G_o.$$

The group of this bundle reduces to $G_{\bar{o}} \times G_{\bar{o}}$, acting on the model fiber $S_o M \times G_o$ by

$$(4.3) \quad (\gamma_0, \gamma_1) \cdot (\zeta, \bar{\gamma}) = (\gamma_0 \zeta, \gamma_0 \bar{\gamma} \gamma_1^{-1}).$$

There is then a double fibration

$$(4.4) \quad \begin{array}{ccc} \mathbf{E} & \xrightarrow{(Z, \Gamma)} & SM \times G \\ (\Xi, H) \downarrow & & \\ SM \times SM & & \end{array}$$

where Ξ, H, Z, Γ are the restricted projections to the respective factors. The left action of $G \times G$ on \mathbf{E} , given by

$$(\gamma_0, \gamma_1) \cdot (\xi, \eta, \zeta, \gamma) := (\gamma_0 \xi, \gamma_1 \eta, \gamma_0 \zeta, \gamma_0 \gamma \gamma_1^{-1}),$$

intertwines the obvious left actions of $G \times G$ on $SM \times SM$ and $SM \times G$ respectively.

4.2.1. *The connecting current.* Each fiber $\mathbf{E}_{\xi, \eta}$ includes the canonical subset

$$\mathbf{C}_{\xi, \eta} := \{(\zeta, \gamma) \in \mathbf{E}_{\xi, \eta} : \zeta \in \overline{\xi, \gamma\eta}\},$$

where $\overline{\xi, \gamma\eta} \subset S_{\pi(\xi)}M$ denotes the set of all points lying on a minimizing geodesic connecting $\xi, \gamma\eta$. Typically this set is a geodesic arc, although if $\xi = \gamma\eta$ it is a point, and if $\xi = -\gamma\eta$ then it is all of $S_{\pi(\xi)}M$. The *interior* $\mathbf{C}_{\xi, \eta}^\circ$, consisting of those elements of $\mathbf{C}_{\xi, \eta}$ for which $\gamma\eta \neq \pm\xi$ and ζ is an interior point of $\overline{\xi, \gamma\eta}$, is then a smooth manifold diffeomorphic to $(0, 1) \times (G_o \setminus \{\gamma : \gamma\bar{o} \neq \pm\bar{o}\})$.

The various $\mathbf{C}_{\xi, \eta}$ may be identified with the model fiber

$$\mathbf{C} := \{(\zeta, \gamma) \in S_o M \times G_o : \zeta \in \overline{\bar{o}, \gamma\bar{o}}\},$$

canonically up to the action (4.3) of $G_{\bar{o}} \times G_{\bar{o}}$. Clearly \mathbf{C} is compact and semialgebraic of dimension $1 + \dim G_o$, hence has finite volume of this dimension. We may take the diffeomorphism

$$(4.5) \quad \begin{aligned} (0, 1) \times (G_o \setminus \{\gamma : \gamma\bar{o} = \pm\bar{o}\}) &\rightarrow \mathbf{C}^\circ \subset S_o M \times G_o, \\ (t, \gamma) &\mapsto (\nu((1-t)\bar{o} + t\gamma\bar{o}), \gamma) \end{aligned}$$

to preserve orientations, thus endowing \mathbf{C} with the structure of an integral current with boundary

$$(4.6) \quad \partial\mathbf{C} = p_{1*}G_o - p_{0*}G_o + K,$$

where $\text{spt } K \subset S_o M \times \{\gamma \in G_o : \gamma\bar{o} = \pm\bar{o}\}$ and $p_0, p_1 : G_o \rightarrow S_o M \times G_o$ are given by

$$(4.7) \quad \begin{aligned} p_1(\gamma) &:= (\gamma\bar{o}, \gamma), \\ p_0(\gamma) &:= (\bar{o}, \gamma). \end{aligned}$$

Lemma 4.3. *Fiber integration over \mathbf{C} yields a well-defined $G \times G$ -equivariant operator*

$$\pi_{\mathbf{C}*} : \Omega^*(\mathbf{E}) \rightarrow \Omega^*(SM \times SM)$$

of degree $-\dim \mathbf{C} = -(1 + \dim G_o)$. In particular, if $\beta \in \Omega^*(SM)^G$ and $d\gamma$ is an invariant volume form for G then there are constants $c_{i,j}$ such that

$$(4.8) \quad \pi_{\mathbf{C}*}(Z^* \beta \wedge \Gamma^* d\gamma) = \sum_{1 \leq i, j \leq N} c_{i,j} \Xi^* \beta_i \wedge H^* \beta_j,$$

where β_1, \dots, β_N constitute a basis for $\Omega^*(SM)^G$.

Proof. Cf. Section 1 of [11]. □

The fiber integration operator gives rise to the following current theoretic construction. Let S, T be currents living in SM . Then there is a well-defined fiber product current $S \times T \times_{\mathcal{E}} \mathbf{C}$ in \mathbf{E} , given in any local trivialization by the corresponding Cartesian product. In particular, if S, T are integral then so is this fiber product. The description of \mathbf{C} above gives rise to the following alternative local expression.

Lemma 4.4. *Let S, T be currents living in SM , and let $W \subset G$ be an open set such that*

$$\gamma \in W \implies \text{spt } S \cap s(\gamma \text{spt } T) = \emptyset = \text{spt } S \cap \gamma \text{spt } T.$$

Then

$$(Z, \Gamma)_*(S \times T \times_{\mathcal{E}} \mathbf{C}) \lfloor \Gamma^{-1}(W) = c_* [(S \times T \times (0, 1) \times_{\mathcal{F}} G_o) \lfloor \Gamma^{-1}(W)]$$

where

$$c(\xi, \eta, t, \gamma) := (\nu((1-t)\xi + t\gamma\eta), \gamma). \quad \square$$

4.3. The formal kinematic formula. We recall the main construction of [11], applied in the WDC context. Let $A, B \in \text{WDC}(M)$. We put for a.e. $\gamma \in G$

$$(4.9) \quad \mathcal{J}(A, B, \gamma) := (-1)^{d \dim G + d - 1} Z_* \langle N(A) \times N(B) \times_{\mathcal{E}} \mathbf{C}, \Gamma, \gamma \rangle \\ + N(A) \lfloor \pi^{-1}(\gamma B) + N(\gamma B) \lfloor \pi^{-1}(A).$$

It will be useful to express the second and third terms of the right hand side of (4.9) in terms of slicing. As usual we denote by Γ the projection maps of the spaces

$$SM \times M \times_{\mathcal{F}} G_o, \quad M \times SM \times_{\mathcal{F}} G_o$$

to G , and put Ξ, H for the respective projections to the SM factors.

Lemma 4.5. *For a.e. $\gamma \in G$*

$$(4.10) \quad \Xi_* \langle N(A) \times B \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle = (-1)^{(d-1) \dim G} N(A) \lfloor \pi^{-1}(\gamma B),$$

$$(4.11) \quad H_* \langle A \times N(B) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle = (-1)^{d+(d-1) \dim G} N(B) \lfloor \pi^{-1}(\gamma^{-1} A)$$

Proof. We prove (4.10), the proof of (4.11) being similar. By [9], Theorem 4.3.8, the left hand side of (4.10) is equal to the restriction to $\pi^{-1}(\gamma B)$ of $\Xi_* \langle N(A) \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle$. Thus we wish to show that this last current equals $(-1)^{(d-1) \dim G} N(A)$.

Let f be an aura for A . From (2.7) we deduce that

$$(4.12) \quad \Xi_* \langle \mathbb{D}(f) \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle = (-1)^{d \dim G} \mathbb{D}(f).$$

Using (2.3) and (2.19) we calculate

$$\begin{aligned} \Xi_* \langle N(A) \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle &= \Xi_* \langle \nu_* \partial(\mathbb{D}(f) \lfloor \pi^{-1}(A)) \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle \\ &= \nu_* \Xi_* \langle \partial(\mathbb{D}(f) \lfloor \pi^{-1}(A)) \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle \\ &= (-1)^{\dim G} \nu_* \Xi_* \partial(\langle \mathbb{D}(f) \lfloor \pi^{-1}(A) \rangle \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma) \\ &= (-1)^{\dim G} \nu_* \partial [\Xi_* \langle \mathbb{D}(f) \times M \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle \lfloor \pi^{-1}(A)] \\ &= (-1)^{(d+1) \dim G} \nu_* \partial [\mathbb{D}(f) \lfloor \pi^{-1}(A)] \quad \text{by (4.12)} \\ &= (-1)^{(d+1) \dim G} N(A). \end{aligned}$$

\square

We now prove a formal version of the kinematic formula (1.2).

Proposition 4.6. *Let $\beta_0 \in \Omega^{d-1,G}(SM)$ be a G -invariant form of degree $d-1$, and let $\beta_1, \dots, \beta_N \in \Omega^{d-1,G}(SM)$ be a basis for the space of all such forms. Then there are constants $c_{ij}, 1 \leq i, j \leq N$, such that for any compact WDC sets $A, B \subset M$ and bounded Borel measurable functions $\phi, \psi : M \rightarrow \mathbb{R}$*

$$(4.13) \quad \begin{aligned} \int_G \int_{\mathcal{J}(A,B,\gamma)} \pi^*(\phi \cdot (\psi \circ \gamma^{-1})) \wedge \beta_0 d\gamma &= \sum_{i,j} c_{ij} \int_{N(A)} \pi^* \phi \wedge \beta_i \cdot \int_{N(B)} \pi^* \psi \wedge \beta_j \\ &+ \int_A \phi \cdot \int_{N(B)} \pi^* \psi \wedge \beta_0 \\ &+ \int_B \psi \cdot \int_{N(A)} \pi^* \phi \wedge \beta_0 \end{aligned}$$

Proof. By the slicing theorem 4.3.2 (1) of [9] and Lemma 4.5, the left hand side of (4.13) may be expressed as the sum

$$\begin{aligned} (-1)^{d \dim G + d + 1} \int_{N(A) \times N(B) \times_{\mathcal{E}} \mathbf{C}} \Gamma^* d\gamma \wedge X^* \phi \wedge Y^* \psi \wedge Z^* \beta_0 &+ \\ &+ (-1)^{(d-1) \dim G} \int_{N(A) \times B \times_{\mathcal{F}} G_o} \Gamma^* d\gamma \wedge \Xi^*(\phi \wedge \beta_0) \wedge Y^* \psi \\ &+ (-1)^{d+(d-1) \dim G} \int_{A \times N(B) \times_{\mathcal{F}} G_o} \Gamma^* d\gamma \wedge X^* \phi \wedge H^*(\psi \wedge \beta_0) \end{aligned}$$

corresponding respectively to the three terms in (4.9), where (as usual) we have abused notation slightly in the labelling of the maps. By Lemma 4.3, the first integral may be expressed as

$$\sum_{1 \leq i, j \leq N} c_{ij} \int_{N(A)} \pi^* \phi \wedge \beta_i \cdot \int_{N(B)} \pi^* \psi \wedge \beta_j.$$

The second and third integrals become

$$\int_{N(A) \times B \times_{\mathcal{F}} G_o} \Xi^*(\phi \wedge \beta_0) \wedge Y^* \psi \wedge \Gamma^* d\gamma + (-1)^d \int_{A \times N(B) \times_{\mathcal{F}} G_o} X^* \phi \wedge H^*(\psi \wedge \beta_0) \wedge \Gamma^* d\gamma$$

which by (2.9), (2.10) yield the other terms on the right hand side of (4.13). \square

4.4. Conclusion of the proof of Theorem B. Together with Proposition 4.6, the following concludes the proof of Theorem B.

Theorem 4.7. *If $A, B \subset M$ are WDC subsets of M then*

$$\mathcal{J}(A, B, \gamma) = N(A \cap \gamma B)$$

for a.e. $\gamma \in G$.

By (2.24), this follows from

Lemma 4.8. *For a.e. $\gamma \in G$*

$$(4.14) \quad \vec{N}(A \cap \gamma B) = z_*(A \cap \gamma B) + m_*([0, \infty) \times \mathcal{J}(A, B, \gamma)).$$

Proof. We claim first that

$$(4.15) \quad \vec{N}(A \cap \gamma B) = (-1)^{d \dim G} \Sigma_* \langle \vec{N}(A) \times \vec{N}(B) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle.$$

To see this, let f, g be auras for A, B respectively, let $C \subset\subset G$ be as in Proposition 4.1, and let $\gamma_0 \in G \setminus C$. Let $W \ni \gamma_0, U \supset A, V \supset B$ be neighborhoods as in

Proposition 4.1. Then Theorem 2.16 and Proposition 2.21 imply that for $\gamma \in W$

$$\begin{aligned}
(-1)^{d \dim G} \vec{N}(A \cap \gamma B) &= (-1)^{d \dim G} \lim_{t \rightarrow \infty} m_{t*}(\mathbb{D}(h_\gamma) \llcorner \pi^{-1}(U \cap \gamma V)) \\
&= \lim_{t \rightarrow \infty} m_{t*} \Sigma_* \langle (\mathbb{D}(f) \llcorner \pi^{-1}(U)) \times (\mathbb{D}(g) \llcorner \pi^{-1}(V)) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle \\
&= \lim_{t \rightarrow \infty} \Sigma_* \langle m_{t*}(\mathbb{D}(f) \llcorner \pi^{-1}(U)) \times m_{t*}(\mathbb{D}(g) \llcorner \pi^{-1}(V)) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle \\
&= \Sigma_* \langle \vec{N}(A) \times \vec{N}(B) \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle.
\end{aligned}$$

Here the third equality is justified by the Remark following Proposition 2.21.

To complete the proof, we show that the right hand side of (4.14) equals the right hand side of (4.15).

From the definition of \vec{N} , the current $\vec{N}(A) \times \vec{N}(B) \times_{\mathcal{F}} G_o$ may be expressed as the sum of the four terms

$$(4.16) \quad z_* A \times z_* B \times_{\mathcal{F}} G_o,$$

$$(4.17) \quad z_* A \times (m_*([0, \infty) \times N(B))) \times_{\mathcal{F}} G_o,$$

$$(4.18) \quad (m_*([0, \infty) \times N(A))) \times z_* B \times_{\mathcal{F}} G_o,$$

$$(4.19) \quad (m_*([0, \infty) \times N(A))) \times (m_*([0, \infty) \times N(B))) \times_{\mathcal{F}} G_o.$$

From (2.7) we deduce that for a.e. $\gamma \in G$

$$\Sigma_* \langle z_* A \times z_* B \times_{\mathcal{F}} G_o, \Gamma, \gamma \rangle = (-1)^{d \dim G} z_*(A \cap \gamma B)$$

and by the same reasoning the Σ images of the Γ slices of (4.17), (4.18) yield, respectively,

$$\begin{aligned}
&(-1)^{d \dim G} m_*([0, \infty) \times N(\gamma B)) \llcorner \pi^{-1}(A), \\
&(-1)^{d \dim G} m_*([0, \infty) \times N(A)) \llcorner \pi^{-1}(\gamma B)
\end{aligned}$$

It remains to show that the same operation, applied to (4.19), yields

$$(-1)^{d \dim G} m_*([0, \infty) \times Z_* \langle N(A) \times N(B) \times_{\mathcal{E}} \mathbf{C}, \Gamma, \gamma \rangle).$$

It is enough to prove this for $\gamma \in W$, where $W \subset G$ is an open set as in Proposition 4.1. In fact we prove the corresponding fact in unsliced form, i.e. that the image of

$$(4.20) \quad (-1)^{d \dim G + d - 1} [0, \infty) \times [(N(A) \times N(B) \times_{\mathcal{E}} \mathbf{C}) \llcorner \Gamma^{-1}(W)]$$

under the map $(t, \xi, \eta, \zeta, \gamma) \mapsto (t\zeta, \gamma)$, corresponding to the right hand side of (4.14), is identical to the image of

$$(4.21) \quad (-1)^{d \dim G} [(m_*([0, \infty) \times N(A))) \times (m_*([0, \infty) \times N(B))) \times_{\mathcal{F}} G_o] \llcorner \Gamma^{-1}(W)$$

under the map $(\Sigma, \Gamma) : (\xi, \eta, \gamma) \mapsto (\xi + \gamma\eta, \gamma)$, which corresponds to (4.15).

By the description (4.5) of the orientation of \mathbf{C} , the image of (4.20) is equal to the image of

$$(-1)^{d \dim G + d - 1} [0, \infty) \times N(A) \times N(B) \times (0, 1) \times_{\mathcal{F}} G_o$$

under $(s, \xi, \eta, t, \gamma) \mapsto (s\nu((1-t)\xi + t\gamma\eta), \gamma)$, or in other words equal to the image of

$$(-1)^{d \dim G} [0, \infty) \times N(A) \times (0, 1) \times N(B) \times_{\mathcal{F}} G_o$$

under $(s, \xi, t, \eta, \gamma) \mapsto (s\nu((1-t)\xi + t\gamma\eta), \gamma)$.

Meanwhile, the image of (4.21) is equal to the image of

$$(-1)^{d \dim G} [0, \infty) \times N(A) \times [0, \infty) \times N(B) \times_{\mathcal{F}} G_o$$

under $(\sigma, \xi, \tau, \eta, \gamma) \mapsto (\sigma\xi + \tau\gamma\eta, \gamma)$. Since $\xi, \gamma\eta$ are linearly independent for $\gamma \in W$, it is easy to see that the map $(s, t) \mapsto (\sigma, \tau)$, where

$$\sigma = \frac{st}{|t\xi + (1-t)\gamma\eta|}$$

$$\tau = \frac{s(1-t)}{|t\xi + (1-t)\gamma\eta|}$$

defines an orientation-preserving diffeomorphism between $(0, \infty) \times (0, 1) \rightarrow (0, \infty) \times (0, \infty)$ (a modification of the standard polar coordinate map $(s, t) \mapsto (s \cos \frac{\pi t}{2}, s \sin \frac{\pi t}{2})$). This completes the proof. \square

5. QUESTIONS AND CONJECTURES

5.1. Structure of WDC sets. As mentioned in the Introduction, the class of WDC sets includes the finite unions of semiconvex sets in general position as studied by [29], and also the boundaries of all convex bodies. However, the variety of geometric behavior exhibited by WDC sets seems clearly much broader than is displayed by these examples. It would be interesting to understand their behavior in more detail.

Here are some specific, and naïve, questions.

- (1) Suppose $A \subset \mathbb{R}^d$ is a WDC set such that the intrinsic volumes $\mu_{k+1}(A) = \dots = \mu_d(A) = 0$. Is it true that A is rectifiable of dimension k ? Using Crofton's formula for WDC sets (cf. [22]), it is not difficult to show that $\mu_k(A)$ equals the k -dimensional integral geometric measure of A . Thus Federer's structure theorem ([9], Theorem 3.3.13) implies that this question will be settled in the affirmative by showing that the k -dimensional Hausdorff measure of A is no greater than $\mu_k(A)$. We expect that this should be resolvable by studying the relation between A and a suitable carrier of $N(A)$.
- (2) Is the distance function from a WDC set A necessarily DC? Is it an aura for A ? This would, in particular, provide us with a DC aura of A which is semiconcave on the complement of A . Another natural question is then whether there is a DC aura for A which is even smooth on A^c .
- (3) We call a set $A \subset \mathbb{R}^d$ **locally WDC** if for every $x \in A$ there is U_x , an open neighborhood of x , and a WDC set A_x such that $A \cap U_x = A_x \cap U_x$. Is a locally WDC set necessarily WDC, for instance, are the k -dimensional DC surfaces (see [22, Section 3.1]) WDC? Note that they are locally WDC by [22, Proposition 3.3].

It is not difficult to see that the main points of the present paper (in particular the kinematic formulas) apply to locally WDC sets as well.

- (4) Is the generic projection of a WDC set again WDC?

5.2. Integral geometric regularity. The class of objects to which kinematic formulas apply seems impossible to describe in classical terms. On the other hand the quest to describe this class in *some* way is irresistible. One might attempt to come to terms with this situation as follows.

Let us say that a class \mathcal{C} of compact subsets of \mathbb{R}^d is an **integral geometric regularity class** (or **igregularity class**) if the following (probably redundant) list of axioms holds:

- (1) Every $A \in \mathcal{C}$ admits a normal cycle $N(A)$.
- (2) \mathcal{C} is stable under diffeomorphisms ϕ of \mathbb{R}^d , with $N(\phi(A)) = \tilde{\phi}_* N(A)$, where $\tilde{\phi}: S^*\mathbb{R}^d \rightarrow S^*\mathbb{R}^d$ is the induced map.

- (3) If $A, B \in \mathcal{C}$ then for a.e. $\gamma \in \overline{SO(d)}$, the intersection $A \cap \gamma B \in \mathcal{C}$, with

$$N(A \cap \gamma B) = \mathcal{J}(A, B, \gamma)$$

where the right hand side is constructed formally from $N(A), N(B)$ as above.

Thus the class of compact semiconvex sets is an irregularity class, and the present paper, together with [22], implies that the same is true of the class of compact WDC subsets of \mathbb{R}^d . By [12], the class of compact subanalytic sets is an “analytical irregularity class,” i.e. the axioms above hold if (2) is replaced by stability under real analytic diffeomorphisms.

Conjecture 5.1. *There exists a unique maximal irregularity class of subsets of \mathbb{R}^d .*

This conjecture may be sharpened as follows. Consider the class \mathcal{N} of compact sets A with the property that there exists a monotone sequence $M_1 \supset M_2 \supset \dots$ of compact smooth domains $\bigcap_n M_n = A$, where the masses of N_{M_n} are bounded by a fixed constant.

Conjecture 5.2. *\mathcal{N} is an irregularity class.*

It is not even known whether every element of \mathcal{N} admits a normal cycle. Simple examples show that it is not possible to obtain $N(A)$ as a subsequential limit of the $N(M_n)$. However, on naive geometric grounds it is natural to suppose that $N(A)$ could be constructed by some kind of pruning procedure from such a subsequential limit.

Conjecture 5.3. *\mathcal{N} is the unique maximal irregularity class of subsets of \mathbb{R}^d .*

However, it is not completely clear whether WDC sets belong to \mathcal{N} .

It also seems possible that the approach of [12] might be made to work in greater generality, as suggested by the following sample statement:

Conjecture 5.4. *Let $f \in MA(\mathbb{R}^d)$. Then $f^{-1}(-\infty, c] \in \mathcal{N}$ for a.e. $c \in \mathbb{R}$.*

5.3. The Weyl tube principle. Until this point we have not mentioned one of the most striking phenomena of integral geometry, often referred to as “the Weyl tube formula”: if $M \subset \mathbb{R}^d$ is a smoothly embedded submanifold then the Federer curvature measures of M are Riemannian invariants. In particular, these measures may be constructed from the structure of M as an inner metric space. It is easy to show that the latter formulation holds also if M is a (not necessarily smooth) compact convex set.

Does it hold also for general WDC sets? This is not known even in the semiconvex case, nor for the case of boundaries of convex sets. The sole exception in the latter framework is the case where the ambient dimension $d = 3$, in which case the boundary of a closed convex set $A \subset \mathbb{R}^3$ is known to be a “manifold of bounded curvature” in the sense of Alexandrov (cf. [24]).

The WDC case presents a still more primitive obstacle: we do not know whether a general connected WDC set A is always a length space, i.e. whether any two points $x, y \in A$ are always joined by a rectifiable path $\subset A$. This would follow if we knew that such A is a Lipschitz neighborhood retract in the sense of [9]; although the proof of Proposition 1.2 of [12] assumes that this is so, this assertion is unjustified at present.

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